Totally nonpositive completions on partial matrices *†

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Abstract

An $n \times n$ real matrix is said to be totally nonpositive if every minor is nonpositive. In this paper, we are interested in totally nonpositive completion problems, that is, does a partial totally nonpositive matrix have a totally nonpositive matrix completion? This problem has, in general, a negative answer. Therefore, we analyze the question: for which labeled graphs G does every partial totally nonpositive matrix, whose associated graph is G, have a totally nonpositive completion? Here we study the mentioned problem when G is a chordal graph or an undirected cycle.

1 Introduction

A *partial matrix* over $\mathbb{R}$ is an $n \times n$ array in which some entries are specified, while the remaining entries are free to be chosen from $\mathbb{R}$. We make the assumption throughout that all diagonal entries are prescribed. A *completion* of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries. A *matrix completion problem* asks which partial matrices have completions with some desired property.

An $n \times n$ partial matrix is said to be *combinatorially symmetric* if the (i, j) entry is specified if and only if the (j, i) entry is.

The specified positions in an $n \times n$ partial matrix $A = (a_{ij})$ can be represented by a graph $G_A = (V, E)$, where the set of vertices V is $\{1, ..., n\}$ and $\{i, j\}$, $i \neq j$, is an edge or arc if the (i, j) entry is specified. G_A is an undirected graph when A is combinatorially symmetric and a directed graph in other case. We omit loops since all diagonal entries are specified.

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An $n \times n$ real matrix $A = (a_{ij})$ is called a *totally nonpositive matrix* if every minor is nonpositive. In particular, this means that $a_{ij} \leq 0$, $i, j \in \{1, 2, \dots, n\}$.

The submatrix of a matrix A, of size $n \times n$, lying in rows α and columns β, $\alpha, \beta \subseteq N = \{1, ..., n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. Therefore, a real matrix A, of size $n \times n$, is a totally nonpositive matrix if $\det A[\alpha|\beta] \geq 0$, for all $\alpha, \beta \subseteq \{1, ..., n\}$ such that $|\alpha| = |\beta|$.

The following simple facts are very useful in the study of totally nonpositive matrices.

Proposition 1.1 *Let $A = (a_{ij})$ be an $n \times n$ totally nonpositive matrix.*

1. *If D is a positive diagonal matrix, then DA and AD are totally nonpositive matrices.*
2. *If D is a positive diagonal matrix, then DAD^{-1} is a totally nonpositive matrix.*
3. *Total nonpositivity is not preserved by permutation similarity.*
4. *If $a_{ii} \neq 0$, for all $i \in \{1, 2, \dots, n\}$, then $a_{ij} < 0$, for all $i, j \in \{1, 2, \dots, n\}$.*
5. *Any submatrix of A is a totally nonpositive matrix.*

The last property of Proposition 1.1 allows us to give the following definition.

Definition 1.1 *A partial matrix is said to be a partial totally nonpositive matrix if every completely specified submatrix is a totally positive matrix.*

Our interest here is in the *totally nonpositive matrix completion problem*, that is, does a partial totally nonpositive matrix have a totally nonpositive matrix completion? The problem has, in general, a negative answer for combinatorially and non-combinatorially symmetric partial matrices.

Example 1.1 (a) Let A be the following non-combinatorially symmetric partial totally nonpositive matrix

$$A = \begin{bmatrix} -1 & -0.1 & -3 \\ x_{21} & -1 & -1 \\ -1 & x_{32} & -1 \end{bmatrix}.$$

A has no totally nonpositive completions since $\det A[\{1, 2\}] \leq 0$ and $\det A[\{2, 3\}|\{1, 3\}] \leq 0$ if and only if $x_{21} \leq -10$ and $x_{21} \geq -1$.

(b) Let B be the following combinatorially symmetric partial totally nonpositive matrix

$$B = \begin{bmatrix} -1 & -1 & x_{13} \\ -2 & 0 & -1 \\ x_{31} & -2 & -1 \end{bmatrix}.$$

Matrix B has no totally nonpositive completions since, for example, $\det B[\{1, 2\}|\{2, 3\}] > 0$ for every value of x_{13}.

Taking into account the example above, the first natural question is: for which graphs G does every partial totally nonpositive matrix, the graph of whose specified entries is G, have a totally nonpositive completion? In this paper we are going to work with combinatorially symmetric partial matrices and therefore with undirected graphs.

Because total nonpositivity is not preserved by permutation similarity we must consider *labeled* graphs, that is, graphs in which the numbering of the vertices is fixed. Note that if A is a partial totally nonpositive matrix with graph G and A has a totally nonpositive completion, then, if B is a partial totally nonpositive matrix whose graph is isomorphic to G, B may not necessarily have a totally nonpositive completion. For example, let

$$A = \begin{bmatrix} -1 & -4 & x_{13} \\ -1 & -1 & -1 \\ x_{31} & -2 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & -1 & -1 \\ -4 & -1 & x_{23} \\ -2 & x_{32} & -1 \end{bmatrix}.$$

Then both A and B are partial totally nonpositive and the graphs of A and B are isomorphic. A totally nonpositive completion of A is given by

$$A_c = \begin{bmatrix} -1 & -4 & -4 \\ -1 & -1 & -1 \\ -2 & -2 & -1 \end{bmatrix}.$$

However, there is no totally nonpositive completion of B since $\det B[\{1,2\}|\{2,3\}] \leq 0$ and $\det B[\{2,3\}|\{1,3\}] \leq 0$ if and only if $x_{23} \geq -1$ and $x_{23} \leq -2$, which is impossible. So, the labeling of the graphs is crucial in determining when there is a totally nonpositive completion.

A *path* is a sequence of edges $\{i_1, i_2\}$, $\{i_2, i_3\}$, ..., $\{i_{k-1}, i_k\}$ in which all vertices are distinct. A *cycle* is a closed path, that is a path in which the first and the last vertices coincide. A graph G is said to be *chordal* if there are no minimal cycles of length greater than or equal to 4 (see [1]). A graph is *complete* if it includes all possible edges between its vertices. A *clique* is an induced subgraph that is complete.

A graph G is said to be *1-chordal graph* [1] if G is a chordal graph in which every pair of maximal cliques C_i, C_j, $C_i \neq C_j$, intersect in at most one vertex (this vertex is called *minimal vertex separator*). If the maximum number of vertices in the intersection between two maximal cliques is p, then the chordal graph is said to be *p-chordal*. A *monotonically labeled 1-chordal graph* (see [2]) is a labeled 1-chordal graph in which the vertices of the maximal cliques are labeled in natural order, that is, for every pair of maximal cliques C_i, C_j, in which $i < j$ and $C_i \cap C_j = \{u\}$, the labeling of the vertices within the two cliques is such that every vertex of $\{v : v \in C_i - u\}$ is labeled less than u and every vertex of $\{w : w \in C_j - u\}$ is labeled greater than u. In analogous way, we say that a cycle, a p-chordal graph, ... is monotonically labeled if its vertices are labeled in natural order.

In Section 2 we analyze the totally nonpositive completion problem for some special cases of chordal graphs. We show that the 1-chordal graph guarantee the existence of the desired completion and we study the 2-chordal case. Finally, in Section 3 we obtain that the "SS-condition" is a necessary and sufficient condition in order to obtain a totally nonpositive completion of a partial totally nonpositive matrix whose graph is a monotonically labeled cycle. The results shown in section 2 and section 3 are similar to ones obtained by Johnson, Kroschel and Lundquist in [2] and by Jordán and Torregrosa in [3], for the totally positive completion problem.

2 Chordal graphs

In this section we study the totally nonpositive completion problem when the associated graph to a partial totally nonpositive matrix is chordal. We will see that the monotonically labeled condition and the non-nullity of the minimal vertex separator are necessary conditions.

Throughout the proof of our main result of this section we use the following classical fact of Frobenius-König (see [4])

Lemma 2.1 *Suppose that A is an $n \times n$ real matrix. If A has a $p \times q$ submatrix of zeros, then A is singular whenever $p + q \geq n + 1$.*

Proposition 2.1 *Let A be an $n \times n$ partial totally nonpositive matrix, whose graph of the specified entries is a monotonically labeled 1-chordal graph with two maximal cliques, such that the entry corresponding to the minimal vertex separator is non-zero. Then, A admits a totally nonpositive completion.*

Proof. We may assume, without loss of generality, that A has the following form

$$A = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{21}^T & -1 & A_{23}^T \\ Y & A_{32} & A_{33} \end{bmatrix},$$

where $A_{12}, A_{21} \in \mathbb{R}^p$, $A_{23}, A_{32} \in \mathbb{R}^q$ and $p + q = n - 1$.

Consider the completion

$$A_c = \begin{bmatrix} A_{11} & A_{12} & -A_{12}A_{23}^T \\ A_{21}^T & -1 & A_{23}^T \\ -A_{32}A_{21}^T & A_{32} & A_{33} \end{bmatrix}$$

of A. We are going to see that A_c is a totally nonpositive matrix.

Let $\alpha, \beta \in \{1, \ldots, n\}$ be such that $|\alpha| = |\beta|$. Let k be the index of the minimal vertex separator. We may define $\alpha_1, \beta_1 \subseteq \{1, \ldots, k-1\}$, $\alpha_2, \beta_2 \subseteq \{k+1, \ldots, n\}$ such that $\alpha - \{k\} = \alpha_1 \cup \alpha_2$ and $\beta - \{k\} = \beta_1 \cup \beta_2$.

Obviously, if $|\alpha| = |\beta| = 1$, $\det A_c[\alpha|\beta] \le 0$. Therefore, we consider α, β such that $|\alpha| = |\beta| > 1$.

Firstly, we study the cases in which at least one of the sets $\alpha_1, \alpha_2, \beta_1$ and β_2 is empty, and we end with the case where all of those sets are non-empty:

(1.) $\alpha_1 \ne \emptyset$ and $\alpha_2 = \emptyset$

(1.1.) $\beta_2 = \emptyset$

In this case, $A_c[\alpha|\beta]$ is a submatrix of

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21}^T & -1 \end{bmatrix}$$

and therefore is totally nonpositive.

(1.2.) $|\beta_2| = 1$

We know that there exist indices $1 \le i_1 < i_2 < \ldots < i_t \le k$ and $1 \le j_1 < j_2 < \ldots < j_t \le n$, with $j_{t-1} \le k < j_t$, such that $\alpha = \{i_1, \ldots, i_t\}$ and $\beta = \{j_1, \ldots, j_t\}$. So, $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1j_1} & -a_{i_1j_2} & \ldots & -a_{i_1j_{t-1}} & -a_{i_1k}a_{kj_t} \\ -a_{i_2j_1} & -a_{i_2j_2} & \ldots & -a_{i_2j_{t-1}} & -a_{i_2k}a_{kj_t} \\ \vdots & \vdots & & \vdots & \vdots \\ -a_{i_tj_1} & -a_{i_tj_2} & \ldots & -a_{i_tj_{t-1}} & -a_{i_tk}a_{kj_t} \end{bmatrix},$$

where each a_{ij} is nonnegative. If $k \in \beta$, $j_{t-1} = k$ and $\det A_c[\alpha|\beta] = 0$. If $k \notin \beta$, $\det A_c[\alpha|\beta] = a_{kj_t} \det A_c[\alpha|(\beta - \beta_2) \cup \{k\}] \le 0$.

(1.3.) $|\beta_2| > 1$

In this case, there exist $1 \le i_1 < i_2 < \ldots < i_t \le k$ and $1 \le j_1 < j_2 < \ldots < j_s < j_{s+1} < \ldots < j_t \le n$, with $j_s \le k < j_{s+1}$, such that $\alpha = \{i_1, \ldots, i_t\}$ and $\beta = \{j_1, \ldots, j_t\}$. So, $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1j_1} & -a_{i_1j_2} & \ldots & -a_{i_1j_s} & -a_{i_1k}a_{kj_{s+1}} & \ldots & -a_{i_1k}a_{kj_t} \\ -a_{i_2j_1} & -a_{i_2j_2} & \ldots & -a_{i_2j_s} & -a_{i_2k}a_{kj_{s+1}} & \ldots & -a_{i_2k}a_{kj_t} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ -a_{i_tj_1} & -a_{i_tj_2} & \ldots & -a_{i_tj_s} & -a_{i_tk}a_{kj_{s+1}} & \ldots & -a_{i_tk}a_{kj_t} \end{bmatrix},$$

where each a_{ij} is nonnegative, so $\det A_c[\alpha|\beta] = 0$.

(2.) $\alpha_1 = \emptyset$ and $\alpha_2 \ne \emptyset$

(2.1.) $\beta_1 = \emptyset$

$A_c[\alpha|\beta]$ is a submatrix of

$$\begin{bmatrix} -1 & A_{23}^T \\ A_{32} & A_{33} \end{bmatrix},$$

and therefore $\det A_c[\alpha|\beta] \le 0$.

(2.2.) $|\beta_1| = 1$

There exist indices $k \leq i_1 < i_2 < \ldots < i_t \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_t \leq n$, with $j_1 < k \leq j_2$, such that $\alpha = \{i_1, \ldots, i_t\}$ and $\beta = \{j_1, \ldots, j_t\}$. So, $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1k}a_{kj_1} & -a_{i_1j_2} & \ldots & -a_{i_1j_{t-1}} & -a_{i_1j_t} \\ -a_{i_2k}a_{kj_1} & -a_{i_2j_2} & \ldots & -a_{i_2j_{t-1}} & -a_{i_2j_t} \\ \vdots & \vdots & & \vdots & \vdots \\ -a_{i_tk}a_{kj_1} & -a_{i_tj_2} & \ldots & -a_{i_tj_{t-1}} & -a_{i_tj_t} \end{bmatrix},$$

where each a_{ij} is nonnegative. If $k \in \beta$, then $j_2 = k$ and $\det A_c[\alpha|\beta] = 0$. If $k \notin \beta$, $\det A_c[\alpha|\beta] = a_{kj_1} \det A_c[\alpha|(\beta - \beta_1) \cup \{k\}] \leq 0$.

(2.3.) $|\beta_1| > 1$

So, there exist $k \leq i_1 < i_2 < \ldots < i_t \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_s < j_{s+1} < \ldots < j_t \leq n$, with $j_s < k \leq j_{s+1}$, such that $\alpha = \{i_1, \ldots, i_t\}$ and $\beta = \{j_1, \ldots, j_t\}$. $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1k}a_{kj_1} & \ldots & -a_{i_1k}a_{kj_s} & -a_{i_1j_{s+1}} & \ldots & -a_{i_1j_{t-1}} & -a_{i_1j_t} \\ -a_{i_2k}a_{kj_1} & \ldots & -a_{i_2k}a_{kj_s} & -a_{i_2j_{s+1}} & \ldots & -a_{i_2j_{t-1}} & -a_{i_2j_t} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -a_{i_tk}a_{kj_1} & \ldots & -a_{i_tk}a_{kj_s} & -a_{i_tj_{s+1}} & \ldots & -a_{i_tj_{t-1}} & -a_{i_tj_t} \end{bmatrix},$$

where each a_{ij} is nonnegative, and therefore $\det A_c[\alpha|\beta] = 0$.

(3.) $\alpha_1 \neq \emptyset$ and $\alpha_2 \neq \emptyset$

(3.1.) $\beta_1 \neq \emptyset$ and $\beta_2 = \emptyset$

(3.1.1.) $|\alpha_2| = 1$

In this case, there exist $1 \leq i_1 < i_2 < \ldots < i_t \leq n$, with $i_{t-1} \leq k < i_t$, and $1 \leq j_1 < j_2 < \ldots < j_t \leq k$ such that $\alpha = \{i_1, \ldots, i_t\}$ and $\beta = \{j_1, \ldots, j_t\}$. So, $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1j_1} & -a_{i_1j_2} & \ldots & -a_{i_1j_t} \\ -a_{i_2j_1} & -a_{i_2j_2} & \ldots & -a_{i_2j_t} \\ \vdots & \vdots & & \vdots \\ -a_{i_{t-1}j_1} & -a_{i_{t-1}j_2} & \ldots & -a_{i_{t-1}j_t} \\ -a_{i_tk}a_{kj_1} & -a_{i_tk}a_{kj_2} & \ldots & -a_{i_tk}a_{kj_t} \end{bmatrix},$$

where each a_{ij} is nonnegative. If $k \in \alpha$, that is, $i_{t-1} = k$, then $\det A_c[\alpha|\beta] = 0$. If $k \notin \alpha$, $\det A_c[\alpha|\beta] = a_{i_tk} \det A_c[(\alpha - \alpha_2) \cup \{k\}|\beta] \leq 0$.

(3.1.2.) $|\alpha_2| > 1$

There exist $1 \leq i_1 < i_2 < \ldots < i_s < i_{s+1} < \ldots < i_t \leq n$, with $i_s \leq k < i_{s+1}$, and

$1 \leq j_1 < j_2 < \ldots < j_t \leq k$ such that $\alpha = \{i_1, \ldots, i_t\}$ and $\beta = \{j_1, \ldots, j_t\}$. So, the submatrix $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1j_1} & -a_{i_1j_2} & \cdots & -a_{i_1j_t} \\ -a_{i_2j_1} & -a_{i_2j_2} & \cdots & -a_{i_2j_t} \\ \vdots & \vdots & & \vdots \\ -a_{i_sj_1} & -a_{i_sj_2} & \cdots & -a_{i_sj_t} \\ -a_{i_{s+1}k}a_{kj_1} & -a_{i_{s+1}k}a_{kj_2} & \cdots & -a_{i_{s+1}k}a_{kj_t} \\ \vdots & \vdots & & \vdots \\ -a_{i_tk}a_{kj_1} & -a_{i_tk}a_{kj_2} & \cdots & -a_{i_tk}a_{kj_t} \end{bmatrix},$$

there each a_{ij} is nonnegative, and therefore $\det A_c[\alpha|\beta] = 0$.

(3.2.) $\beta_1 = \emptyset$ and $\beta_2 \neq \emptyset$

(3.2.1.) $|\alpha_1| = 1$
There exist $1 \leq i_1 < i_2 < \ldots < i_t \leq n$, with $i_1 < k \leq i_2$, and $k \leq j_1 < j_2 < \ldots < j_t \leq n$ such that $\alpha = \{i_1, \ldots, i_t\}$ and $\beta = \{j_1, \ldots, j_t\}$. The submatrix $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1k}a_{kj_1} & -a_{i_1k}a_{kj_2} & \cdots & -a_{i_1k}a_{kj_t} \\ -a_{i_2j_1} & -a_{i_2j_2} & \cdots & -a_{i_2j_t} \\ \vdots & \vdots & & \vdots \\ -a_{i_tj_1} & -a_{i_tj_2} & \cdots & -a_{i_tj_t} \end{bmatrix},$$

with $a_{ij} \geq 0$ for all i and j. So, if $k \in \alpha$, $\det A_c[\alpha|\beta] = 0$. If $k \notin \alpha$, $\det A_c[\alpha|\beta] = a_{i_1k} \det A_c[(\alpha - \alpha_1) \cup \{k\}|\beta] \leq 0$.

(3.2.2.) $|\alpha_1| > 1$
There exist $1 \leq i_1 < \ldots < i_s < i_{s+1} < \ldots < i_t \leq n$, with $i_s < k \leq i_{s+1}$, and $k \leq j_1 < j_2 < \ldots < j_t \leq n$ such that $\alpha = \{i_1, \ldots, i_t\}$ e $\beta = \{j_1, \ldots, j_t\}$. Then, $A_c[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1k}a_{kj_1} & -a_{i_1k}a_{kj_2} & \cdots & -a_{i_1k}a_{kj_t} \\ \vdots & \vdots & & \vdots \\ -a_{i_sk}a_{kj_1} & -a_{i_sk}a_{kj_2} & \cdots & -a_{i_sk}a_{kj_t} \\ -a_{i_{s+1}j_1} & -a_{i_{s+1}j_2} & \cdots & -a_{i_{s+1}j_t} \\ \vdots & \vdots & & \vdots \\ -a_{i_tj_1} & -a_{i_tj_2} & \cdots & -a_{i_tj_t} \end{bmatrix},$$

with $a_{ij} \geq 0$, for all i, j. So, $\det A_c[\alpha|\beta] = 0$.

In order to finish the proof we need to study the case

(3.3.) $\beta_1 \neq \emptyset$ and $\beta_2 \neq \emptyset$

We know that there exist $i_1, i_2, \ldots, i_s, i_{s+1}, \ldots, i_t, j_1, j_2, \ldots, j_r, j_{r+1}, \ldots,\ j_p \in \{1, \ldots, n\}$ such that $i_1 < \ldots < i_s < i_{s+1} < \ldots < i_t, j_1 < \ldots < j_r < j_{r+1} < \ldots < j_p$ and $\alpha_1 = \{i_1, \ldots, i_s\}, \alpha_2 = \{i_{s+1}, \ldots, i_t\}, \beta_1 = \{j_1, \ldots, j_r\}$ and $\beta_2 = \{j_{r+1}, \ldots, j_p\}$.

Consider the following cases:

(3.3.1.) $k \in \alpha$ and $k \in \beta$

In this case, $t = p$ and $A_c\,[\alpha|\beta]$ has the form

$$\begin{bmatrix} -a_{i_1 j_1} & \ldots & -a_{i_1 j_r} & -a_{i_1 k} & -a_{i_1 k} a_{k j_{r+1}} & \ldots & -a_{i_1 k} a_{k j_t} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -a_{i_s j_1} & \ldots & -a_{i_s j_r} & -a_{i_s k} & -a_{i_s k} a_{k j_{r+1}} & \ldots & -a_{i_s k} a_{k j_t} \\ -a_{k j_1} & \ldots & -a_{k j_r} & -1 & -a_{k j_{r+1}} & \ldots & -a_{k j_t} \\ -a_{i_{s+1} k} a_{k j_1} & \ldots & -a_{i_{s+1} k} a_{k j_r} & -a_{i_{s+1} k} & -a_{i_{s+1} j_{r+1}} & \ldots & -a_{i_{s+1} j_t} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -a_{i_t k} a_{k j_1} & \ldots & -a_{i_t k} a_{k j_r} & -a_{i_t k} & -a_{i_t j_{r+1}} & \ldots & -a_{i_t j_t} \end{bmatrix},$$

where each a_{ij} is nonnegative.

(3.3.1.1.) $|\beta_1| - |\alpha_1| \geq 1$

For each $l \in \{s+1, \ldots, t\}$ such that $a_{i_l k} \neq 0$, we add to the $(l+1)$-th row of $A_c\,[\alpha|\beta]$ the $(s+1)$-th row multiplied by $-a_{i_l k}$. The matrix $\tilde{A}$ obtained has the form

$$\tilde{A} = \left[\begin{array}{c|c} A_c[\alpha_1 \cup \{k\}|\beta_1 \cup \{k\}] & A_c[\alpha_1 \cup \{k\}|\beta_2] \\ \hline 0 & B \end{array}\right].$$

Note that $\tilde{A}$ has a zero submatrix of size $|\alpha_2| \times (|\beta_1| + 1)$. By applying Lemma 2.1, $\det \tilde{A} = 0$ if $|\alpha_2| + |\beta_1| + 1 \geq |\alpha| + 1$. But,

$$\begin{aligned} |\alpha_2| + |\beta_1| + 1 \geq |\alpha| + 1 \quad &\Longleftrightarrow \quad |\alpha_2| + |\beta_1| + 1 \geq |\alpha_1| + |\alpha_2| + 1 + 1 \\ &\Longleftrightarrow \quad |\beta_1| - |\alpha_1| \geq 1. \end{aligned}$$

So, $\det \tilde{A} = 0$ and therefore $\det A_c\,[\alpha|\beta] = 0$.

(3.3.1.2.) $|\beta_1| - |\alpha_1| = 0$

In this case, we can obtain the same matrix $\tilde{A}$ of the previous case. It is easy to proof that $\det \tilde{A} = \det A_c[\alpha_1 \cup \{k\}|\beta_1 \cup \{k\}] \det B$. Now, we are going to prove that $\det B$ is nonnegative. Consider the submatrix $A_c[\{k\} \cup \alpha_2|\{k\} \cup \beta_2]$

$$\begin{bmatrix} -1 & -a_{k j_{r+1}} & \ldots & -a_{k j_t} \\ -a_{i_{s+1} k} & -a_{i_{s+1} j_{r+1}} & \ldots & -a_{i_{s+1} j_t} \\ \vdots & \vdots & & \vdots \\ -a_{i_t k} & -a_{i_t j_{r+1}} & \ldots & -a_{i_t j_t} \end{bmatrix}.$$

For each $l \in \{s+1,\ldots,t\}$ such that $a_{i_lk} \neq 0$, we add to the $(l-s+1)$-th row of $A_c[\{k\}\cup\alpha_2|\{k\}\cup\beta_2]$ the first row multiplied by $-a_{i_lk}$. Then, we obtain the matrix

$$C = \left[\begin{array}{c|c} -1 & A_c[\{k\}|\beta_2] \\ \hline 0 & B \end{array}\right].$$

Since $\det C = \det A_c[\{k\}\cup\alpha_2|\{k\}\cup\beta_2] \leq 0$ and $\det C = -1 \times \det B$, we can assure that $\det B \geq 0$. Therefore $\det A_c[\alpha|\beta] \leq 0$.

(3.3.1.3.) $|\alpha_1| - |\beta_1| \geq 1$
In analogous way to the case (3.3.1.1.) we obtain that $\det A_c[\alpha|\beta] = 0$.

(3.3.2.) $k \in \alpha$ and $k \notin \beta$
In this case, $t = p-1$ and $A_c\,[\alpha|\beta]$ has the form

$$\begin{bmatrix}
-a_{i_1j_1} & \ldots & -a_{i_1j_r} & -a_{i_1k}a_{kj_{r+1}} & \ldots & -a_{i_1k}a_{kj_p} \\
\vdots & & \vdots & \vdots & & \vdots \\
-a_{i_sj_1} & \ldots & -a_{i_sj_r} & -a_{i_sk}a_{kj_{r+1}} & \ldots & -a_{i_sk}a_{kj_p} \\
-a_{kj_1} & \ldots & -a_{kj_r} & -a_{kj_{r+1}} & \ldots & -a_{kj_p} \\
-a_{i_{s+1}k}a_{kj_1} & \ldots & -a_{i_{s+1}k}a_{kj_r} & -a_{i_{s+1}j_{r+1}} & \ldots & -a_{i_{s+1}j_p} \\
\vdots & & \vdots & \vdots & & \vdots \\
-a_{i_tk}a_{kj_1} & \ldots & -a_{i_tk}a_{kj_r} & -a_{i_tj_{r+1}} & \ldots & -a_{i_tj_p}
\end{bmatrix},$$

where $a_{ij} \geq 0$ for all i and j.

In analogous way to the case (3.3.1.) we obtain the result analyzing the subcases

(3.3.2.1.) $|\beta_1| - |\alpha_1| \geq 2$

(3.3.2.2.) $|\beta_1| - |\alpha_1| = 1$

(3.3.2.3.) $|\alpha_1| - |\beta_1| \geq 1$

(3.3.2.4.) $|\beta_1| - |\alpha_1| = 0$

(3.3.3.) $k \notin \alpha$ and $k \in \beta$
In this case, $t = p+1$ and $A_c\,[\alpha|\beta]$ has the form

$$\begin{bmatrix}
-a_{i_1j_1} & \ldots & -a_{i_1j_r} & -a_{i_1k} & -a_{i_1k}a_{kj_{r+1}} & \ldots & -a_{i_1k}a_{kj_p} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
-a_{i_sj_1} & \ldots & -a_{i_sj_r} & -a_{i_sk} & -a_{i_sk}a_{kj_{r+1}} & \ldots & -a_{i_sk}a_{kj_p} \\
-a_{i_{s+1}k}a_{kj_1} & \ldots & -a_{i_{s+1}k}a_{kj_r} & -a_{i_{s+1}k} & -a_{i_{s+1}j_{r+1}} & \ldots & -a_{i_{s+1}j_p} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
-a_{i_tk}a_{kj_1} & \ldots & -a_{i_tk}a_{kj_r} & -a_{i_tk} & -a_{i_tj_{r+1}} & \ldots & -a_{i_tj_p}
\end{bmatrix},$$

where $a_{ij} \geq 0$ for all i,j.

Analyzing the subcases

(3.3.3.1.) $|\beta_1| - |\alpha_1| \geq 1$

(3.3.3.2.) $|\beta_1| - |\alpha_1| = 0$

(3.3.3.3.) $|\alpha_1| - |\beta_1| \geq 2$

(3.3.3.4.) $|\alpha_1| - |\beta_1| = 0$

we obtain the desired result.

(3.3.4.) $k \notin \alpha$ and $k \notin \beta$

Now, we have that $t = p$ and $A_c\,[\alpha|\beta]$ has the following form

$$\begin{bmatrix} -a_{i_1j_1} & \dots & -a_{i_1j_r} & -a_{i_1k}a_{kj_{r+1}} & \dots & -a_{i_1k}a_{kj_t} \\ \vdots & & \vdots & \vdots & & \vdots \\ -a_{i_sj_1} & \dots & -a_{i_sj_r} & -a_{i_sk}a_{kj_{r+1}} & \dots & -a_{i_sk}a_{kj_t} \\ -a_{i_{s+1}k}a_{kj_1} & \dots & -a_{i_{s+1}k}a_{kj_r} & -a_{i_{s+1}j_{r+1}} & \dots & -a_{i_{s+1}j_t} \\ \vdots & & \vdots & \vdots & & \vdots \\ -a_{i_tk}a_{kj_1} & \dots & -a_{i_tk}a_{kj_r} & -a_{i_tj_{r+1}} & \dots & -a_{i_tj_t} \end{bmatrix},$$

where $a_{ij} \geq 0$ for all i and j.

(3.3.4.1.) $|\beta_1| - |\alpha_1| \geq 2$

First, suppose that $a_{i_{s+1}k} = 0$. Since for all $j > k$ the submatrix

$$A\,[\{k, i_{s+1}\}|\{k, j\}] = \begin{bmatrix} -1 & -a_{kj} \\ 0 & -a_{i_{s+1}j} \end{bmatrix}.$$

is totally nonpositive, then $a_{i_{s+1}j} = 0$. Therefore, $\det A_c\,[\alpha|\beta] = 0$.

Now, we consider that $a_{i_{s+1}k} \neq 0$. For each $l \in \{s+2, \dots, t\}$ such that $a_{i_lk} \neq 0$, we add to the l-th row of $A_c\,[\alpha|\beta]$ the $(s+1)$-th row multiplied by $-a_{i_lk}a_{i_{s+1}k}^{-1}$ and we obtain

$$\tilde{A} = \left[\begin{array}{c|c} A_c[\alpha_1 \cup \{i_{s+1}\}|\beta_1] & A_c[\alpha_1 \cup \{i_{s+1}\}|\beta_2] \\ \hline 0 & B \end{array}\right],$$

Note that $\tilde{A}$ has a zero submatrix of size $(|\alpha_2| - 1) \times |\beta_1|$ since

$$\begin{aligned} |\alpha_2| - 1 + |\beta_1| \geq |\alpha| + 1 \quad &\Longleftrightarrow \quad |\alpha_2| - 1 + |\beta_1| \geq |\alpha_1| + |\alpha_2| + 1 \\ &\Longleftrightarrow \quad |\beta_1| - |\alpha_1| \geq 2, \end{aligned}$$

then $\det A_c[\alpha|\beta] = 0$.

(3.3.4.2.) $|\beta_1| - |\alpha_1| = 1$
If $a_{i_{s+1}k} = 0$ we obtain the result in the same form as in the previous case.

If $a_{i_{s+1}k} \neq 0$ consider the matrix $\tilde{A}$ obtained in the case above. Note that the submatrices $A_c[\alpha_1 \cup \{i_{s+1}\}|\beta_1]$ and B are square matrices and then $\tilde{A}$ is block triangular matrix. Now, consider the square matrix

$$A_c\left[\alpha_2|\{k\} \cup \beta_2\right] = \left[\begin{array}{c|c} -a_{i_{s+1}k} & A_c\left[\{i_{s+1}\}|\beta_2\right] \\ \hline A_c\left[\alpha_2 - \{i_{s+1}\}|\{k\}\right] & A_c[\alpha_2 - \{i_{s+1}\}|\beta_2] \end{array}\right].$$

For each $l \in \{s+2, \ldots, t\}$ such that $a_{i_l k} \neq 0$, we add to the $(l-s+1)$-th row of $A_c\left[\alpha_2|\{k\} \cup \beta_2\right]$ the first row multiplied by $-a_{i_l k} a^{-1}_{i_{s+1}k}$. We obtain the matrix

$$C = \left[\begin{array}{c|c} -a_{i_{s+1}k} & A_c\left[\{i_{s+1}\}|\beta_2\right] \\ \hline 0 & B \end{array}\right].$$

Since $\det C = \det A_c\left[\alpha_2|\{k\} \cup \beta_2\right] \leq 0$ and $a_{i_{s+1}k} \neq 0$, we may assure that $\det B \geq 0$ and therefore $\det \tilde{A} = \det A_c[\alpha_1 \cup \{i_{s+1}\}|\beta_1] \det B \leq 0$.

(3.3.4.3.) $|\alpha_1| - |\beta_1| \geq 2$
It is easy to see it is a symmetric case to (3.3.4.1.) and therefore we obtain the result in analogous way.

(3.3.4.4.) $|\alpha_1| - |\beta_1| = 1$
It is a symmetric case to (3.3.4.2.).

(3.3.4.5.) $|\beta_1| - |\alpha_1| = 0$
Consider the following partition of the submatrix $A_c[\alpha \cup \{k\}|\beta \cup \{k\}]$ of A_c

$$A_c[\alpha \cup \{k\}|\beta \cup \{k\}] = \left[\begin{array}{c|c|c} B & u & -uv^T \\ \hline w^T & -1 & v^T \\ \hline -zw^T & z & C \end{array}\right],$$

where $B = A_c[\alpha_1|\beta_1]$, $u = A_c[\alpha_1|\{k\}]$, $w^T = A_c[\{k\}|\beta_1]$, $v^T = A_c[\{k\}|\beta_2]$, $z = A_c[\alpha_2|\{k\}]$ and $C = A_c[\alpha_2|\beta_2]$. Note that

$$A_c[\alpha|\beta] = \left[\begin{array}{c|c} B & -uv^T \\ \hline -zw^T & C \end{array}\right].$$

Firstly, we suppose that B is nonsingular. As

$$\det \begin{bmatrix} B & u \\ w^t & -1 \end{bmatrix} \leq 0 \iff \det B \times \det(-1 - w^T B^{-1} u) \leq 0,$$

we have $\lambda = -w^T B^{-1} u \geq 1$.

In addition
$$\det \begin{bmatrix} -1 & v^T \\ z & C \end{bmatrix} \leq 0 \iff -1 \times \det(C - z(-1)v^T) \leq 0,$$
and we have $\det(C + zv^T) \geq 0$.

Taking into account that
$$\begin{aligned} \det \begin{bmatrix} B & -uv^T \\ -zw^t & C \end{bmatrix} &= \det B \det(C - zw^t B^{-1} uv^T) \\ &= \det B \det(C + \lambda zv^T), \end{aligned}$$
we need to prove that $\det(C + \lambda zv^T) \geq 0$.

We denote $C = (c_{ij})_{i,j=1}^m$ and $zv^T = (b_{ij})_{i,j=1}^m$. Then, $\det(C + \gamma zv^T) = \det C + \gamma M$, where $\gamma \in \mathbb{R}$ and
$$M = \det \begin{bmatrix} b_{11} & c_{12} & \dots & c_{1m} \\ b_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & c_{m2} & \dots & c_{mm} \end{bmatrix} + \ldots + \det \begin{bmatrix} c_{11} & \dots & c_{1m-1} & b_{1m} \\ c_{21} & \dots & c_{2m-1} & b_{2m} \\ \vdots & & \vdots & \vdots \\ c_{m1} & \dots & c_{mm-1} & b_{mm} \end{bmatrix}.$$

If $\gamma = 1$, we have $\det(C + \gamma zv^T) = \det(C + zv^T) \geq 0$. Then, $M \geq -\det C \geq 0$.

If $\gamma = \lambda$, we have $\det(C + \gamma zv^T) = \det(C + \lambda zv^T) = \det C + \lambda M$. So,
$$\det(C + \lambda zv^T) \geq 0 \iff \lambda M \geq -\det C.$$
Since $\lambda \geq 1$ and $M \geq 0$,
$$\lambda M \geq M \geq -\det C,$$
therefore $\det A_c[\alpha|\beta] \leq 0$.

Finally, we suppose that B is singular. Then, there exists $h \in \{1, \ldots, s\}$ such that
$$-a_{ij_h} = \sum_{l \in (\{j_1, \ldots, j_s\} - \{j_h\})} (-\xi_l a_{il}),$$
for all $i \in \{i_1, \ldots, i_s\}$.

Consider the submatrix $A_c[\alpha_1 \cup \{k\}|\beta_1 \cup \{k\}]$ whose determinant, nonpositive, is

$$\det \begin{bmatrix} -a_{i_1j_1} & -a_{i_1j_2} & \dots & \sum\limits_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (-\xi_l a_{i_1l}) & \dots & -a_{i_1j_s} & -a_{i_1k} \\ -a_{i_2j_1} & -a_{i_2j_2} & \dots & \sum\limits_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (-\xi_l a_{i_2l}) & \dots & -a_{i_2j_s} & -a_{i_2k} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ -a_{i_sj_1} & -a_{i_sj_2} & \dots & \sum\limits_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (-\xi_l a_{i_sl}) & \dots & -a_{i_sj_s} & -a_{i_sk} \\ -a_{kj_1} & -a_{kj_2} & \dots & -a_{kj_h} & \dots & -a_{kj_s} & -1 \end{bmatrix}$$

$$\begin{aligned} &= \det \begin{bmatrix} -a_{i_1j_1} & -a_{i_1j_2} & \dots & 0 & \dots & -a_{i_1j_s} & -a_{i_1k} \\ -a_{i_2j_1} & -a_{i_2j_2} & \dots & 0 & \dots & -a_{i_2j_s} & -a_{i_2k} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ -a_{i_sj_1} & -a_{i_sj_2} & \dots & 0 & \dots & -a_{i_sj_s} & -a_{i_sk} \\ -a_{kj_1} & -a_{kj_2} & \dots & -a_{kj_h} + \sum\limits_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (\xi_l a_{kl}) & \dots & -a_{kj_s} & -1 \end{bmatrix} \\ &= (-1)^{s+1+h} \times \left(-a_{kj_h} + \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (\xi_l a_{kl})\right) \times \det A_c[\alpha_1|(\beta_1 - \{j_h\}) \cup \{k\}]. \end{aligned}$$

Then,

$$(-1)^{s+1+h} \times \left(-a_{kj_h} + \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (\xi_l a_{kl})\right) \geq 0.$$

On the other hand, consider the totally nonpositive submatrix $A_c[(\alpha_1 - \{i_1\}) \cup \{k\}|\beta_1]$ of A_c, whose determinant is

$$\det \begin{bmatrix} -a_{i_2j_1} & -a_{i_2j_2} & \dots & \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (-\xi_l a_{i_2l}) & \dots & -a_{i_2j_s} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{i_sj_1} & -a_{i_sj_2} & \dots & \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (-\xi_l a_{i_sl}) & \dots & -a_{i_sj_s} \\ -a_{kj_1} & -a_{kj_2} & \dots & -a_{kj_h} & \dots & -a_{kj_s} \end{bmatrix}$$

$$\begin{aligned} = \quad & \det \begin{bmatrix} -a_{i_2j_1} & -a_{i_2j_2} & \dots & 0 & \dots & -a_{i_2j_s} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{i_sj_1} & -a_{i_sj_2} & \dots & 0 & \dots & -a_{i_sj_s} \\ -a_{kj_1} & -a_{kj_2} & \dots & -a_{kj_h} + \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (\xi_l a_{kl}) & \dots & -a_{kj_s} \end{bmatrix} \\ = \quad & (-1)^{s+h} \times \left(-a_{kj_h} + \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (\xi_l a_{kl})\right) \times \det A_c[(\alpha_1 - \{i_1\})|(\beta_1 - \{j_h\})]. \end{aligned}$$

Then,

$$(-1)^{s+h} \times \left(-a_{kj_h} + \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (\xi_l a_{kl})\right) \geq 0.$$

So

$$a_{kj_h} = \sum_{l\in(\{j_1,\dots,j_s\}-\{j_h\})} (\xi_l a_{kl}).$$

Now, we can conclude that the h-th column of

$$\begin{bmatrix} B \\ -zw^T \end{bmatrix}$$

is a linear combination of the remaining columns. Therefore,

$$r\left(\begin{bmatrix} B \\ -zw^T \end{bmatrix}\right) < |\alpha_1|$$

and we have

$$r\left(\begin{bmatrix} B & -uv^T \\ -zw^T & C \end{bmatrix}\right) < |\alpha_1| + |\alpha_2| = \alpha.$$

Then, $\det A_c[\alpha|\beta] = 0$. □

We can extend this result in the following way.

Theorem 2.1 *Let G be an undirected connected monotonically labeled 1-chordal graph. Then any partial totally nonpositive matrix, whose graph of the specified entries is G and whose entries corresponding to the minimal vertex separator are non-zero, has a totally nonpositive matrix completion.*

Proof. Let A be a partial totally nonpositive matrix, the graph of whose specified entries is G. The proof is by induction on the number p of maximal cliques in G. For $p = 2$ we obtain the desired completion by applying Proposition 2.1. Suppose that the result is true for a monotonically labeled 1-chordal graph with $p-1$ maximal cliques and we are going to prove it for p maximal cliques.

Let G_1 be the subgraph induced by two maximal cliques with a common vertex. By applying Proposition 2.1 to the submatrix A_1 of A, the graph of whose specified entries is G_1, and by replacing the obtained completion A_{1_c} in A, we obtain a partial totally nonpositive matrix whose associated graph is a monotonically labeled 1-chordal graph with $p-1$ maximal cliques. The induction hypothesis allows us to obtain the result. □

As we saw in Example 1.1 (b), we can not omit the condition of having non-zero minimal vertex separators. On the other hand, in the following example we present a partial totally nonpositive matrix, whose associated graph is a non-monotonically labeled 1-chordal graph with two maximal cliques, with non-zero minimal vertex separator, that has no totally nonpositive completions.

Example 2.1 Consider the partial totally nonpositive matrix

$$A = \begin{bmatrix} -1 & x_{12} & -2 & x_{14} \\ x_{21} & -1 & -4 & -8 \\ -1 & -5 & -1 & -2 \\ x_{41} & -10 & -2 & -1 \end{bmatrix},$$

whose associated graph is a non-monotonically labeled 1-chordal graph with two maximal cliques.

A has no totally nonpositive completions since $\det A[\{1,2\}|\{2,3\}] = -4x_{12} - 2 \leq 0$ and $\det A[\{1,3\}|\{1,2\}] = 5 + x_{12} \leq 0$ if and only if $x_{12} \geq -0.5$ and $x_{12} \leq -5$, which is impossible.

The totally nonpositive completion problem for partial matrices whose associated graph is a p-chordal graph, $p \geq 2$, is yet an open problem. We have obtained partial results for matrices of size 4×4 whose associated graph is a monotonically labeled 2-chordal graph and such that their main diagonal entries are non-zero.

The study of this problem for this particular case illustrates the difficulties of working in the more general case.

We will assume, throughout the rest of this section, that all the diagonal entries are non-zero.

Lemma 2.2 *Let A be a partial totally nonpositive matrix, whose graph is a monotonically labeled 2-chordal graph and such that its minimal vertex separator is singular. Then, there exists a totally nonpositive completion A_c of A.*

Proof. We may assume, without loss of generality, that A has the following form:

$$A = \begin{bmatrix} -1 & -1 & -a_{13} & x_{14} \\ -a_{21} & -1 & -1 & -a_{24} \\ -a_{31} & -1 & -1 & -1 \\ x_{41} & -a_{42} & -a_{43} & -1 \end{bmatrix}.$$

Since A is a partial totally nonpositive matrix $\det A[\{1,2,3\}]$ and $\det A[\{2,3,4\}]$ are nonpositive, then $a_{31} = a_{21}$ or $a_{13} = 1$, and $a_{42} = a_{43}$ or $a_{24} = 1$. In both cases it is easy to prove that

$$A_c = \begin{bmatrix} -1 & -1 & -a_{13} & -a_{13}a_{24} \\ -a_{21} & -1 & -1 & -a_{24} \\ -a_{31} & -1 & -1 & -1 \\ -a_{31}a_{42} & -a_{42} & -a_{43} & -1 \end{bmatrix}$$

is a totally nonpositive matrix completion of A. □

Now, we are going to study the case in which the minimal vertex separator is nonsingular.

Lemma 2.3 *Consider the following partial totally nonpositive matrix*

$$A = \begin{bmatrix} -1 & -1 & -a_{13} & x_{14} \\ -a_{21} & -1 & -1 & -a_{24} \\ -a_{31} & -a_{32} & -1 & -1 \\ x_{41} & -a_{42} & -a_{43} & -1 \end{bmatrix},$$

where each a_{ij} is positive and $a_{32} > 1$. Then, the partial matrix B obtained from A by replacing the entry x_{41} by $-a_{31}a_{42}a_{32}^{-1}$, is also a partial totally nonpositive matrix.

Proof. It is easy to prove that all totally specified minors, of size 2×2, are nonpositive.

Since A is a partial totally nonpositive matrix, we know that $\det B[\{1,2,3\}]$ and $\det B[\{2,3,4\}]$ are nonpositive.

For the remaining totally specified submatrices, of size 3×3, we have

$$\begin{aligned}
\det B[\{1,2,4\}|\{1,2,3\}] &= a_{42}a_{32}^{-1}\det B[\{1,2,3\}] + (a_{42} - a_{32}a_{43})(1 - a_{21})a_{32}^{-1} \leq 0, \\
\det B[\{1,3,4\}|\{1,2,3\}] &= (a_{31} - a_{32})(a_{32}a_{43} - a_{42})a_{32}^{-1} \leq 0, \\
\det B[\{2,3,4\}|\{1,2,3\}] &= (a_{42} - a_{32}a_{43})(a_{21}a_{32} - a_{31})a_{32}^{-1} \leq 0, \\
\det B[\{2,3,4\}|\{1,2,4\}] &= (a_{42} - a_{32})(a_{21}a_{32} - a_{31})a_{32}^{-1} \leq 0, \\
\det B[\{2,3,4\}|\{1,3,4\}] &= a_{31}a_{32}^{-1}\det B[\{2,3,4\}] + (a_{31} - a_{21}a_{32})(1 - a_{43})a_{32}^{-1} \leq 0.
\end{aligned}$$

Then, B is a partial totally nonpositive matrix. □

Lemma 2.4 *Let B be the following partial totally nonpositive matrix*

$$B = \begin{bmatrix} -1 & -1 & -a_{13} & x_{14} \\ -a_{21} & -1 & -1 & -a_{24} \\ -a_{31} & -a_{32} & -1 & -1 \\ -a_{31}a_{42}a_{32}^{-1} & -a_{42} & -a_{43} & -1 \end{bmatrix},$$

where each a_{ij} is positive and $a_{32} > 1$. Consider the partition

$$B = \left[\begin{array}{c|cc|c} -1 & -1 & -a_{13} & x_{14} \\ \hline -a_{21} & -1 & -1 & -a_{24} \\ -a_{31} & -a_{32} & -1 & -1 \\ \hline -a_{31}a_{42}a_{32}^{-1} & -a_{42} & -a_{43} & -1 \end{array}\right] = \left[\begin{array}{c|c|c} -1 & A_{12} & x_{14} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline -a_{31}a_{42}a_{32}^{-1} & A_{32} & -1 \end{array}\right].$$

Then,

$$B_c = \left[\begin{array}{c|c|c} -1 & A_{12} & A_{12}A_{22}^{-1}A_{23} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline -a_{31}a_{42}a_{32}^{-1} & A_{32} & -1 \end{array}\right]$$

is a totally nonpositive completion of B.

Proof. It is easy to prove that all minors of size 2×2 are nonpositive. Since B is a partial totally nonpositive matrix, we have

$$\begin{aligned} \det B_c\,[\alpha|\{1,2,3\}] &\leq 0 \quad \forall \alpha \in \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}, \\ \det B_c\,[\{2,3,4\}|\beta] &\leq 0 \quad \forall \beta \in \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}. \end{aligned}$$

On the other hand, it is easy to see that

$$\det B_c = \frac{\det B\,[\{1,2,3\}]\det B\,[\{2,3,4\}]}{\det B\,[\{2,3\}]} \leq 0.$$

Now, for $\alpha = \{1,2,3\}$, we obtain

$$\begin{aligned} \det B_c\,[\alpha|\{1,2,4\}] &= (a_{32}-1)^{-1}(a_{24}a_{32}-1)\det B\,[\alpha] \leq 0, \\ \det B_c\,[\alpha|\{1,3,4\}] &= (a_{32}-1)^{-1}(a_{24}-1)\det B\,[\alpha] \leq 0, \\ \det B_c\,[\alpha|\{2,3,4\}] &= 0. \end{aligned}$$

For $\alpha = \{1,2,4\}$,

$$\begin{aligned} \det B_c\,[\alpha] &= a_{42}a_{32}^{-1}\det B_c\,[\{1,2,3\}|\{1,2,4\}] + a_{32}^{-1}(a_{42}-a_{32})(1-a_{21}) \leq 0, \\ \det B_c\,[\alpha|\{1,3,4\}] &= a_{42}a_{32}^{-1}\det B_c\,[\{1,2,3\}|\{1,3,4\}] \\ &\quad +(a_{32}-a_{32}^2)^{-1}(a_{32}a_{43}-a_{42})(a_{24}-1)(1-a_{21}) \\ &\quad +(1-a_{32})^{-1}(1-a_{13}a_{21})\det B\,[\{2,3,4\}] \leq 0, \\ \det B_c\,[\alpha|\{2,3,4\}] &= (a_{32}-1)^{-1}(a_{13}-1)\det B_c\,[\{2,3,4\}] \leq 0. \end{aligned}$$

Finally, for $\alpha = \{1,3,4\}$,

$$\begin{aligned} \det B_c\,[\alpha|\{1,2,4\}] &= a_{32}^{-1}(a_{32}-a_{42})(a_{31}-a_{32}) \leq 0, \\ \det B_c\,[\alpha] &= a_{31}a_{32}^{-1}\det B_c\,[\alpha|\{2,3,4\}] + a_{32}^{-1}(a_{31}-a_{32})(1-a_{43}) \leq 0, \\ \det B_c\,[\alpha|\{2,3,4\}] &= (a_{32}-1)^{-1}(a_{13}a_{32}-1)\det B\,[\{2,3,4\}] \leq 0. \end{aligned}$$

Therefore, B_c is a totally nonpositive completion of matrix B. □

The previous Lemmas allow us to give the following result.

Theorem 2.2 *Let A be a partial totally nonpositive matrix, of size 4×4, whose associated graph is a monotonically labeled 2-chordal graph. Then, there exists a totally nonpositive completion A_c of A.*

In this result we can not omit the hypothesis of monotonically labeled as we can see in the following example:

Example 2.2 Consider the partial totally nonpositive matrix

$$A = \begin{bmatrix} -1 & x_{12} & -1 & -2 \\ x_{21} & -1 & -1 & -2 \\ -1.5 & -2 & -1 & -1 \\ -3.5 & -4.5 & -2 & -1 \end{bmatrix},$$

whose graph is a non-monotonically labeled 2-chordal graph.

A has no totally nonpositive completions since

$$\begin{aligned} \det A[\{1,2\}|\{2,3\}] \le 0 &\iff x_{12} \ge -1, \\ \det A[\{1,3\}|\{1,2\}] \le 0 &\iff x_{12} \le -4/3 \end{aligned}$$

which is impossible.

3 Cycles

In this section we study the totally nonpositive completion problem for partial matrix whose associated graph is a cycle. In general, the mentioned problem has a negative answer for cycles that are or not monotonically labeled, as we will see in the next example.

A cycle $G = (V, E)$, with $V = \{1, \ldots, n\}$, is said to be *monotonically labeled* if $E = \big\{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}, \{n,1\}\big\}$. In this section we consider that all specified entries are non-zero, since in other case the results, in general, do not hold.

Example 3.1 (a) Consider the partial totally nonpositive matrix

$$A = \begin{bmatrix} -1 & -1 & x_{13} & -0.5 \\ -2 & -1 & -1 & x_{24} \\ x_{31} & -2 & -1 & -1 \\ -8 & x_{42} & -2 & -1 \end{bmatrix},$$

whose associated graph is a monotonically labeled cycle. A has no totally nonpositive completions since

$$\begin{aligned} \det A\,[\{1,2\}|\{2,3\}] \le 0 &\iff x_{13} \le -1, \\ \det A\,[\{1,3\}|\{3,4\}] \le 0 &\iff x_{13} \ge -0.5. \end{aligned}$$

(b) Now, consider the following partial totally nonpositive matrix whose associated graph is a cycle that is not monotonically labeled

$$B = \begin{bmatrix} -1 & x_{12} & -2 & -4 \\ x_{21} & -1 & -4 & -6 \\ -1 & -5 & -1 & x_{34} \\ -2 & -10 & x_{43} & -1 \end{bmatrix}.$$

In analogous way to (a), B has no totally nonpositive completions since

$$\begin{aligned} \det B\,[\{1,2\}|\{2,3\}] \leq 0 \quad &\Longleftrightarrow \quad x_{12} \geq -0.5, \\ \det B\,[\{1,3\}|\{1,2\}] \leq 0 \quad &\Longleftrightarrow \quad x_{12} \leq -5. \end{aligned}$$

In this section we only study the mentioned problem when the associated graph to the $n \times n$ partial totally nonpositive matrix is a monotonically labeled cycle. We can assume, without loss of generality, that this type of matrices have the form:

$$A = \begin{bmatrix} -1 & -a_{12} & x_{13} & \cdots & x_{1n-1} & -a_{1n} \\ -a_{21} & -1 & -a_{23} & \cdots & x_{2n-1} & x_{2n} \\ x_{31} & -a_{32} & -1 & \cdots & x_{3n-1} & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n-11} & x_{n-12} & x_{n-13} & \cdots & -1 & -a_{n-1n} \\ -a_{n1} & x_{n2} & x_{n3} & \cdots & -a_{nn-1} & -1 \end{bmatrix}.$$

Definition 3.1 *Let A be a partial totally nonpositive matrix, of size $n \times n$, whose associated graph is a monotonically labeled cycle. We say that A satisfies the "SS-diagonal condition" if*

$$a_{12}a_{23}\cdots a_{n-1n} \leq a_{1n} \quad \textit{and} \quad a_{nn-1}a_{n-1n-2}\cdots a_{21} \leq a_{n1}.$$

Lemma 3.1 *Let A be a partial totally nonpositive matrix, of size 4×4, whose graph is a monotonically labeled cycle. There exists a totally nonpositive completion A_c of A if and only if A satisfies the SS-diagonal condition.*

Proof. Suppose that there exists a totally nonpositive completion A_c of A,

$$A_c = \begin{bmatrix} -1 & -a_{12} & -c_{13} & -a_{14} \\ -a_{21} & -1 & -a_{23} & -c_{24} \\ -c_{31} & -a_{32} & -1 & -a_{34} \\ -a_{41} & -c_{42} & -a_{43} & -1 \end{bmatrix}$$

where all c_{ij} is positive. From $\det A_c\,[\{1,2\}|\{2,3\}] \leq 0$ and $\det A_c\,[\{1,3\}|\{3,4\}] \leq 0$, we get $a_{14} \geq a_{12}a_{23}a_{34}$. An analogous reasoning for $\det A_c\,[\{3,4\}|\{2,3\}]$ and $\det A_c\,[\{2,4\}|\{1,2\}]$ gives the condition $a_{41} \geq a_{21}a_{32}a_{43}$.

For the sufficiency we take the completion of A

$$A_c = \begin{bmatrix} -1 & -a_{12} & -a_{12}a_{23} & -a_{14} \\ -a_{21} & -1 & -a_{23} & -a_{14}a_{12}^{-1} \\ -a_{41}a_{43}^{-1} & -a_{32} & -1 & -a_{34} \\ -a_{41} & -a_{32}a_{43} & -a_{43} & -1 \end{bmatrix}.$$

By using the SS-diagonal condition, it is easy to show the nonpositivity of all minors of A_c. □

This result can be generalized to matrices of size $n \times n$, $n > 4$.

Theorem 3.1 *Let A be a partial totally nonpositive matrix, of size $n \times n$, $n \geq 4$, whose graph is a monotonically labeled cycle. There exists a totally nonpositive completion A_c of A if and only if A satisfies the SS-diagonal condition.*

Proof. The proof is by induction on n. If $n = 4$ we apply Lemma 3.1. Suppose that the result holds for matrices of size $(n-1) \times (n-1)$ and we are going to see it for $n \times n$ matrices.

Suppose that there exists a totally nonpositive completion of A

$$A_c = \begin{bmatrix} -1 & -a_{12} & -c_{13} & \dots & -c_{1n-1} & -a_{1n} \\ -a_{21} & -1 & -a_{23} & \dots & -c_{2n-1} & -c_{2n} \\ -c_{31} & -a_{32} & -1 & \dots & -c_{3n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -c_{n-11} & -c_{n-12} & -c_{n-13} & \dots & -1 & -a_{n-1n} \\ -a_{n1} & -c_{n2} & -c_{n3} & \dots & -a_{nn-1} & -1 \end{bmatrix},$$

with $c_{ij} > 0$, for all i, j. Then, $A_c[\{2, \dots, n\}]$ is a totally nonpositive completion of the partial totally nonpositive matrix

$$\begin{bmatrix} -1 & -a_{23} & \dots & x_{2n-1} & -c_{2n} \\ -a_{32} & -1 & \dots & x_{3n-1} & x_{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{n-12} & x_{n-13} & \dots & -1 & -a_{n-1n} \\ -c_{n2} & x_{n3} & \dots & -a_{nn-1} & -1 \end{bmatrix}.$$

By induction hypothesis we know that $a_{23}a_{34}\cdots a_{n-1n} \leq c_{2n}$ and $a_{32}a_{43}\cdots a_{nn-1} \leq c_{n2}$. From $\det A_c[\{1,2\}|\{2,n\}] = a_{12}c_{2n} - a_{1n} \leq 0$ and $\det A_c[\{2,n\}|\{1,2\}] = a_{21}c_{n2} - a_{n1} \leq 0$, we can conclude the SS-diagonal condition.

Conversely, we suppose that the SS-diagonal condition is satisfied. It is easy to prove that

$$\bar{A} = \begin{bmatrix} -1 & -a_{12} & x_{13} & \dots & x_{1n-1} & x_{1n} \\ -a_{21} & -1 & -a_{23} & \dots & x_{2n-1} & -a_{1n}a_{12}^{-1} \\ x_{31} & -a_{32} & -1 & \dots & x_{3n-1} & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n-11} & x_{n-12} & x_{n-13} & \dots & -1 & -a_{n-1n} \\ x_{n1} & -a_{n1}a_{21}^{-1} & x_{n3} & \dots & -a_{nn-1} & -1 \end{bmatrix},$$

is a partial totally nonpositive matrix.

Consider the submatrix $\bar{A}[\{2,\ldots,n\}] = (-b_{ij})_{i,j=1}^{n-1}$. We can observe that it is a partial totally nonpositive matrix, whose associate graph is a monotonically labeled cycle with $n-1$ vertices, and satisfying the SS-diagonal condition. By induction hypothesis there exists a totally nonpositive completion

$$\bar{A}[\{2,\ldots,n\}]_c = \begin{bmatrix} -1 & c_{12} \\ c_{21}^T & C_{22} \end{bmatrix}.$$

Now, consider the following partial totally nonpositive matrix

$$\bar{\bar{A}} = \begin{bmatrix} -1 & -a_{12} & x_{1n} \\ -a_{21} & -1 & c_{12} \\ x_{n1} & c_{21}^T & C_{22} \end{bmatrix}.$$

Note that its associated graph is a monotonically labeled 1-chordal graph with two maximal cliques. By Proposition 2.1, there exists a totally nonpositive completion A_c of $\bar{\bar{A}}$ whose elements in positions $(1,n)$ and $(n,1)$ are $-a_{1n}$ and $-a_{n1}$, respectively. Therefore, A_c is the desired completion of A. $\square$

Finally, we are going to see that the previous result does not hold when some diagonal entry are zero.

Example 3.2 Consider the partial totally nonpositive matrix

$$A = \begin{bmatrix} -1 & -3 & x_{13} & -1 \\ -4 & -2 & -2 & x_{24} \\ x_{31} & 0 & 0 & 0 \\ -1 & x_{42} & -0.1 & -0.1 \end{bmatrix},$$

whose associated graph is a monotonically labeled cycle. A satisfies the SS-diagonal condition, however it does not admit a totally nonpositive completion since

$$\begin{aligned} \det A[\{1,2\}|\{2,4\}] \leq 0 &\iff x_{24} \geq -2/3, \\ \det A[\{2,4\}|\{3,4\}] \leq 0 &\iff x_{24} \leq -2. \end{aligned}$$

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